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COMPUTING $\psi(x)$

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ABSTRACT. Let Λ denote the *Von Mangoldt* function and $\psi(x) = \sum_{n \le x} \Lambda(n)$. We describe an elementary method for computing isolated values of $\psi(x)$. The complexity of the algorithm is $O(x^{2/3}(\log\log x)^{1/3})$ time and $O(x^{1/3}(\log\log x)^{2/3})$ space. A table of values of $\psi(x)$ for x up to 10^{15} is included, and some times of computation are given.

1. INTRODUCTION

One of the oldest problems in mathematics is to compute the exact number of primes $\le x$, denoted by $\pi(x)$. This can be achieved by at least three completely different methods:

- any method (like the sieve of *Eratosthenes*) which finds all primes $\le x$ and therefore cannot be achieved with less than about $\frac{x}{\log x}$ operations (by the Chebychev Theorem).
- the *Meissel-Lehmer* combinatorial method, which uses sieve identities, computes $\pi(x)$ in $O(\frac{x^{2/3}}{\log^2 x})$ time and $O(x^{1/3}\log^3 x \log\log x)$ space using the improvements of *Lagarias*, *Miller* and *Odlyzko* [5] and *Deléglise-Rivat* [2].
- the *Lagarias-Odlyzko* analytic method [6], based on numerical integration of certain integral transforms of the *Riemann* ζ-function, for computing $\pi(x)$ using $O_\varepsilon(x^{1/2+\varepsilon})$ time and $O_\varepsilon(x^{1/4+\varepsilon})$ space for each $\varepsilon > 0$.

The Von Mangoldt function $\Lambda(n)$ is defined by $\Lambda(n) = \ln p$ if $n = p^\alpha$ with p a prime number and α an integer ≥ 1, and $\Lambda(n) = 0$ otherwise.

The Prime Number Theorem ($\pi(x) \sim \frac{x}{\log x}$) is well known to be equivalent to $\psi(x) \sim x$. Moreover $\Lambda(n)$ satisfies combinatorial identities based on *Dirichlet* convolutions. Therefore people usually try to replace the characteristic function of the primes by $\Lambda(n)$ when possible. Most proofs of the Prime Number Theorem involve $\Lambda(n)$. Taking advantage of the structure of $\Lambda(n)$, we can efficiently compute $\psi(x)$ in a much simpler manner than $\pi(x)$.

We note that the *Lagarias-Odlyzko* [6] analytic method could also be adapted for computing $\psi(x)$ in $O_\varepsilon(x^{1/2+\varepsilon})$ time. To our knowledge, nobody has tried to compute $\pi(x)$ or $\psi(x)$ using their method yet.

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2. Vaughan's combinatorial identity

It is a classical method (Hoheisel [4], Vinogradov [8]) to transform a sum of the form $\sum_n \Lambda(n) f(n)$ into a few multiple sums

$$\sum_{n_1,\dots,n_k} a_1(n_1)\cdots a_k(n_k) f(n_1\cdots n_k),$$

where $n_1,\dots,n_k$ satisfies multiplicative conditions.

Vaughan has given an elegant formulation of the method in [7], which was enhanced by Heath-Brown [3].

Consider the combinatorial identity $-\frac{\zeta'}{\zeta} = F - \zeta' G - \zeta F G + (\frac{1}{\zeta} - G)(-\zeta' - \zeta F)$, where $G(s) = \sum_{n\le u} \frac{\mu(n)}{n^s}$ and $F(s) = \sum_{n\le u} \frac{\Lambda(n)}{n^s}$.

On picking out the coefficient of n^{-s} on each side we obtain

$$\sum_{n\le x} \Lambda(n) f(n) = S_1(x,u) + S_2(x,u) - S_3(x,u) - S_4(x,u),$$

with

$$\begin{aligned}
S_1(x,u) &= \sum_{n\le u} \Lambda(n) f(n),\\
S_2(x,u) &= \sum_{\substack{m\le u\\ mn\le x}} \mu(m) \ln n f(mn),\\
S_3(x,u) &= \sum_{\substack{l\le u\\ m\le u\\ lmn\le x}} \mu(l)\Lambda(m) f(lmn),\\
S_4(x,u) &= \sum_{\substack{u<m\le x\\ u<n\le x\\ mn\le x}} \Lambda(m) \sum_{\substack{d\mid n\\ d\le u}} \mu(d) f(mn).
\end{aligned}$$

3. Computing $\psi(x)$

We apply the Vaughan identity with $f(n) = 1$ for all n.

In order to compute $S_1(x,u)$, $S_2(x,u)$, $S_3(x,u)$, $S_4(x,u)$, suppose that we have a tabulation of

- $\mu(m)$ for $1 \le m \le u$,
- $\Lambda(n)$ for $1 \le n \le u$.

$S_1(x,u)$ can be easily computed in $O(u)$ time.

For computing $S_2(x,u)$ we simply write

$$S_2(x,u) = \sum_{m\le u} \mu(m) \sum_{n\le x/m} \ln n,$$

which can be computed in $O(u)$ time, using the Euler-MacLaurin method.

The computation of $S_3(x,u)$ is also elementary, using the formula

$$S_3(x,u) = \sum_{l\le u} \sum_{m\le u} \mu(l)\Lambda(m) \left\lfloor \frac{x}{lm} \right\rfloor,$$

which can be computed in $O(u^2)$ time.

It remains to compute $S_4(x,u)$. We have

$$S_4(x,u) = \sum_{l\le u}\mu(l) \sum_{\frac{u}{l}<m\le \frac{x}{ul}} \left(\psi(\frac{x}{lm}) - \psi(u)\right).$$

We remark that if $m > \sqrt{x/l}$ we will have $\frac{x}{lm} \le \sqrt{x/l}$ and the expression $\psi(\frac{x}{lm})$ will remain constant for several consecutive values of m. More precisely, for any fixed $l \le u$ and $k \le \sqrt{x/l}$, let us denote by $N(x,u,l,k)$ the number of m's such that $\sqrt{x/l} < m \le \frac{x}{ul}$ and $\lfloor \frac{x}{lm} \rfloor = k$. We then have

$$\begin{aligned} S_4(x,u) &= \sum_{l\le u}\mu(l) \sum_{\frac{u}{l}<m\le\sqrt{x/l}} \left(\psi(\frac{x}{lm}) - \psi(u)\right) \\ &\quad + \sum_{l\le u}\mu(l) \sum_{k\le\sqrt{x/l}} \left(\psi(k) - \psi(u)\right) N(x,u,l,k). \end{aligned}$$

For any fixed $l \le u$ and $k \le \sqrt{x/l}$ the computation of $N(x,u,l,k)$ can be done in $O(1)$ time.

Hence the computation of $S_4(x,u)$ needs $O(\sum_{l\le u}\sqrt{x/l}) = O(\sqrt{xu})$ time, provided we have a tabulation of $\psi(t)$ for $t \le \frac{x}{u}$.

Conclusion: the time complexity of the method is

$$O(\frac{x}{u}\log\log x + u^2 + \sqrt{xu}).$$

Choosing $u = x^{1/3}(\log\log x)^{2/3}$ we obtain the expected algorithm in $O(x^{2/3}(\log\log x)^{1/3})$ time.

For the space complexity, we work by blocks of size $O(u)$ during the computation of $S_4(x,u)$. This can be done without changing the time complexity (see [1] for more details).

The computations were done using a HP 730 workstation using HP C++ and HP 128 bits emulating floating point arithmetic. A 128 bit log function was missing and has therefore been implemented.

The precision of all computations was 33 decimal digits and the results are presented in Table 1 and Table 2 with 21 decimal digits. That means that even for the computation of $\psi(10^{15})$ which needed about 10^{10} operations, we have removed 12 digits to ensure a safe result.

Using emulated arithmetic instead of hardware arithmetic was a severe inconvenience in terms of speed (we loose a factor of 10), if we compare with the computation of $M(10^{15})$ in [1] (115674 seconds).

The computation of $\pi(10^{15})$ in [2] running in $O(\frac{x^{2/3}}{\log^2 x})$ time is much faster (4179 seconds), thanks to the $\log^2 x$ factor, but the method is much more sophisticated.

TABLE 1. Values of $\psi(x)$ for $10^6 \le x \le 10^{10}$

x	$\psi(x)$	Time (s)
$1e+06$	999586.597495632922033	1.9
$2e+06$	2000115.04620704883194	2.7
$3e+06$	2999999.97999224824973	3.3
$4e+06$	3999490.85679656995798	3.8
$5e+06$	5000971.14022810153042	4.4
$6e+06$	5999649.57769000335617	4.9
$7e+06$	7000575.18641502034942	5.2
$8e+06$	8000121.73320157678229	5.8
$9e+06$	9000850.24888020485237	6.2
$1e+07$	9998539.40334597536635	6.6
$2e+07$	20000600.0251592610472	9.9
$3e+07$	30000704.2820934588192	12.8
$4e+07$	40001480.2149926336305	15.2
$5e+07$	50001207.3445023684082	17.4
$6e+07$	59999308.9772123490642	19.6
$7e+07$	70000783.2023729056695	21.4
$8e+07$	79997966.4586902581049	23.1
$9e+07$	89995860.2769185707641	25.0
$1e+08$	99998242.7966267823416	27.0
$2e+08$	199997027.504552593271	42.0
$3e+08$	299999378.662858843880	54.6
$4e+08$	400002778.057726641750	65.2
$5e+08$	500006989.938817115113	75.0
$6e+08$	600001708.590910478782	85.9
$7e+08$	700004314.549532205866	94.7
$8e+08$	799998546.590393988537	103
$9e+08$	899984812.936571262951	111
$1e+09$	1000001595.99042758043	119
$2e+09$	1999987159.49785559537	188
$3e+09$	2999993292.11099204139	243
$4e+09$	4000010994.99711695725	301
$5e+09$	4999978986.63843391783	345
$6e+09$	6000009612.90884384952	387
$7e+09$	7000003157.58512856840	433
$8e+09$	7999982212.86641692741	470
$9e+09$	8999991956.06404171841	513
$1e+10$	10000042119.8334736147	542

TABLE 2. Values of $\psi(x)$ for $10^{10} \le x \le 10^{15}$

x	$\psi(x)$	Time (s)
$1e+10$	10000042119.8334736147	542
$2e+10$	19999966102.3907942572	862
$3e+10$	29999948420.7708689779	1131
$4e+10$	40000011887.3168320418	1369
$5e+10$	49999955855.4610665034	1590
$6e+10$	60000021580.8714738616	1793
$7e+10$	70000038604.9247522381	1994
$8e+10$	80000005722.4617696008	2173
$9e+10$	89999948906.7797648192	2347
$1e+11$	100000058456.430302189	2527
$2e+11$	200000148773.856006802	3990
$3e+11$	299999977708.641374443	5249
$4e+11$	399999741196.670035169	6344
$5e+11$	499999820953.584593629	7362
$6e+11$	600000033739.152232002	8332
$7e+11$	699999845411.761649322	9202
$8e+11$	800000037979.274743740	10015
$9e+11$	899999777231.876070005	10831
$1e+12$	1000000040136.76545665	11698
$2e+12$	2000000182627.33596499	18519
$3e+12$	2999999566058.80822946	24237
$4e+12$	4000000386475.41118430	29205
$5e+12$	5000000327315.75362324	34042
$6e+12$	5999999744293.47085658	38252
$7e+12$	6999999601425.70691002	42600
$8e+12$	8000000713529.43266003	46258
$9e+12$	8999999446379.56396960	50290
$1e+13$	10000000171997.1232250	53848
$2e+13$	19999999625767.6651778	85328
$3e+13$	30000001040718.2137042	111172
$4e+13$	39999999893274.9689501	135183
$5e+13$	49999999652324.2650673	156036
$6e+13$	59999998082525.3515850	177031
$7e+13$	70000002724370.2485641	195814
$8e+13$	79999999149546.6793392	213221
$9e+13$	89999999033193.6246454	233010
$1e+14$	100000000618647.548001	248385
$2e+14$	199999997677127.254625	394900
$3e+14$	300000004090602.822282	514659
$4e+14$	400000002371843.685660	627765
$5e+14$	499999996459514.248704	726220
$6e+14$	600000008190785.239956	818700
$7e+14$	699999998433148.184857	904060
$8e+14$	799999993059175.785429	988647
$9e+14$	899999991484841.192344	1074297
$1e+15$	999999997476930.507683	1153859

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